TREE AUTOMATA AND SEPARABLE SETS OF INPUT VARIABLES

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Abstract. We introduce the separable sets of variables for trees and tree automata. If a set Y of input variables is inseparable for a tree and an automaton then there a non empty family of distributive sets of Y. It is shown that if a tree t has "many" inseparable sets with respect to a tree automaton $\mathcal{A}$ then there is an effective way to reduce the complexity of $\mathcal{A}$ when running on t.

1. Introduction

The consideration that finite automata may be viewed as unary algebras is attributed to J.Büchi and J.Wright [10]. In many papers trees were defined as terms. Investigations on regular and context-free tree grammars dated back to the 60-th. Tree automata are designed in the context of circuit verification and logic programming. Since the end of 70's tree automata have been used as powerful tools in program verification. There are many results connecting properties of programs or type systems or rewrite systems with automata [3, 4].
The algebraic theory of terms was created and developed upto the equational theory in the work of A.Malc'ev, G.Grätzer etc.[1, 7, 5].
The theory of essential variables and separable sets for discrete functions was created and developed by S.Jablonsky, A.Salomaa, K.Chimev etc.[2, 6, 8]. The results obtained here are very useful for analysis and synthesis of functional schemes and circuits.
The present paper is a continuation and generalization of the results in [9] which are borderline cases of these fields of theoretical computer science and mathematics.

2. Preliminaries

Let $\mathcal{F}$ be any finite set, the elements of which are called *operation symbols*. Let $\tau : \mathcal{F} \to N$ be a mapping into the non negative integers; for $f \in \mathcal{F}$, the number $\tau(f)$ will denote the *arity* of the operation symbol f. The pair $(\mathcal{F}, \tau)$ is called *type* or *signature*. If it is obvious what the set $\mathcal{F}$ is, we will write "*type* τ". The set of symbols of arity p is denoted by $\mathcal{F}_p$. Elements of arity $0, 1, \dots, p$ respectively are called *constants(nullary), unary,...,p-ary* symbols. We assume that $\mathcal{F}_0 \neq \emptyset$.

Definition 2.1. Let $X = \{x_1, x_2 \dots, \}$ be a set of distinct objects called variables, and let τ be a type with the set of operation symbols $\mathcal{F} = \cup_{i\geq 0}\mathcal{F}_i = (f_i)_{i\in I}$. The set $W_\tau(X)$ of *terms of type* τ with variables from X is the smallest set such that
(i) $X \cup \mathcal{F}_0 \subseteq W_\tau(X)$;
(ii) if f is $n-$ary operation symbol and $t_1, \dots, t_n$ are terms then the "string" $f(t_1 \dots t_n)$ is a term.

Note that terms are also called *trees.*
Let t be a term then the set $Var(t)$ consisting of these elements of X which occur in t is called the set of *input variables (or variables)* for this term.
The *depth* of a tree t is defined in the following inductive way:
(i) If $t \in X \cup \mathcal{F}_0$ then $Depth(t) = 0$;
(ii) If $t = f(t_1, \ldots, t_n)$ then $Depth(t) = max\{Depth(t_1), \ldots, Depth(t_n)\} + 1$.
If $t = f(t_1, \ldots, t_n)$ then $t,\ t_1, \ldots, t_n$ are *subterms (subtrees)* of t and all subtrees of $t_1, \ldots, t_n$ are subtrees of t, too.
Thus we define a partial order relation in the set of all terms $W_\tau(X)$. We denote by $\trianglelefteq$ the subterm ordering, i.e. we write $t \trianglelefteq t'$ if t is a subterm of t'. We denote $t \lhd t'$ if $t \trianglelefteq t'$ and $t \neq t'$. A chain of subterms $t_1 \lhd t_2 \lhd \ldots \lhd t_k$ is called *strong* if there does not exist a term s such that $t_j \lhd s \lhd t_{j+1}$ for some $j \in \{1, \ldots, k-1\}$.
Let $t, t' \in W_\tau(X)$ and $t_1 \trianglelefteq t$. We denote by $t(t_1 \leftarrow t')$ the term which is obtained by substituting in t simultaneously t' for each occurrence of t_1 as a subterm of t.

3. Finite Tree Automata and Separable Sets of Input Variables

Definition 3.1. A *finite tree automaton* over $\mathcal{F}$ and X is a tuple $\mathcal{A} = \langle Q, \mathcal{F}, X, Q_f, \Delta\rangle$ where, $\mathcal{F}$ and X are sets of operational symbols and variables, Q is a finite set of states, $Q_f \subseteq Q$ is a set of final states and Δ is the set of transition rules, $\Delta = \{\Delta_0, \Delta_1, \ldots, \Delta_n\}$, where $\Delta_0 : \mathcal{F}_0 \to Q$, and $\Delta_i : \mathcal{F}_i \times Q^i \to Q,\ i = 1, \ldots, n$ are mappings. In this paper we will consider complete and deterministic automata only i.e. Δ_i is a total function for each $i = 0, 1, \ldots, n$.

Let $Y \subseteq X$ be a set of variables and $\gamma : Y \to \mathcal{F}_0$ be a function which assigns nullary operation symbols (constants) to each input variable from Y. The function γ is called *assignment* on the set of inputs Y. The set of such assignments will be denoted by $Ass(Y, \mathcal{F}_0)$.

Let $t \in W_\tau(X)$, $\gamma \in Ass(Y, \mathcal{F}_0)$ and $Y = \{x_1, \ldots, x_m\}$. The term $t(x_1 \leftarrow \gamma(x_1), \ldots, x_m \leftarrow \gamma(x_m))$ will be denoted by $\gamma(t)$. We will definitely assume that if $x_i \in Y \setminus Var(t)$ then $t(x_i \leftarrow \gamma(x_i)) = t$ for each $\gamma \in Ass(Y, \mathcal{F}_0)$.

It is clear that if $Y \cap Z = \emptyset$, $\gamma_1 \in Ass(Y, \mathcal{F}_0)$ and $\gamma_2 \in Ass(Z, \mathcal{F}_0)$ then $\gamma_1(\gamma_2(t)) = \gamma_2(\gamma_1(t))$.

Let $\gamma \in Ass(X, \mathcal{F}_0)$. The automaton $\mathcal{A} = \langle Q, \mathcal{F}, X, Q_f, \Delta\rangle$ runs on t and γ. It starts at leaves of t and moves downwards, associating along the run a resulting state with each subterm inductively:
(i) If $Depth(t) = 0$ then the automaton $\mathcal{A}$ associates the state $q \in Q$ with t, where $q = \Delta_0(\gamma(x_i))$ if $t = x_i \in X$ and $q = \Delta_0(f_0)$ if $t = f_0 \in \mathcal{F}_0$.
(ii) Let $Depth(t) \geq 1$. If $t = f(t_1, \ldots, t_n)$ and the states $q_1, \ldots, q_n$ are associated with the subterms $t_1, \ldots, t_n$ then the automaton $\mathcal{A}$ associates the state q with t, where $q = \Delta_n(f, q_1, \ldots, q_n)$.
A term t in $W_\tau(X)$ is accepted by an automaton $\mathcal{A} = \langle Q, \mathcal{F}, X, Q_f, \Delta\rangle$ if there exists an assignment γ such that when running on t and γ the automaton $\mathcal{A}$ associates with t a final state $q \in Q_f$.

When $\mathcal{A}$ associates the state q with a tree s, and an assignment $\gamma \in Ass(X, \mathcal{F}_0)$ we will write $\mathcal{A}(\gamma, s) = q$.

Definition 3.2. An input variable $x_i \in Var(t)$ is called *essential* for t and $\mathcal{A}$ if there exist two assignments $\gamma_1, \gamma_2 \in Ass(X, \mathcal{F}_0)$ such that $\gamma_1(x_j) = \gamma_2(x_j)$, for each variable $x_j, x_j \neq x_i$ and $\mathcal{A}(\gamma_1, t) \neq \mathcal{A}(\gamma_2, t)$.

The set of all essential inputs for t and $\mathcal{A}$ is denoted by $Ess(t, \mathcal{A})$. The input variables from $Var(t) \setminus Ess(t, \mathcal{A})$ are called *fictive* for t and $\mathcal{A}$.

Lemma 1. Let $f_0 \in \mathcal{F}_0$. If $x_i \notin Ess(t, \mathcal{A})$ then

$$\mathcal{A}(\gamma, t) = \mathcal{A}(\gamma, t(x_i \leftarrow f_0))$$

for each $\gamma \in Ass(X, \mathcal{F}_0)$.

Proof. Suppose the lemma is false and let $\gamma_0 \in Ass(X, \mathcal{F}_0)$ be an assignment such that $\mathcal{A}(\gamma_0, t) \neq \mathcal{A}(\gamma_0, t(x_i \leftarrow f_0))$. Consider the assignment $\gamma_1 \in Ass(X, \mathcal{F}_0)$ defined by $\gamma_1(x) = f_0$ if $x = x_i$, and $\gamma_1(x) = \gamma_0(x)$ if $x \neq x_i$. Hence $\mathcal{A}(\gamma_1, t) = \mathcal{A}(\gamma_0, t(x_i \leftarrow f_0)) \neq \mathcal{A}(\gamma_0, t)$, i.e. $x_i \in Ess(t, \mathcal{A})$. A contradiction. ■

Lemma 2. Let $t, s \in W_\tau(X)$. If $x_i \notin Ess(t, \mathcal{A})$ and for each $q \in Q$ there exists $f_0 \in \mathcal{F}_0$ such that $\Delta_0(f_0) = q$ then

$$\mathcal{A}(\gamma, t) = \mathcal{A}(\gamma, t(x_i \leftarrow s))$$

for each $\gamma \in Ass(X, \mathcal{F}_0)$.

Proof. Suppose that the lemma is false and let $\gamma_0 \in Ass(X, \mathcal{F}_0)$ be such assignment that $\mathcal{A}(\gamma_0, t) \neq \mathcal{A}(\gamma_0, t(x_i \leftarrow s))$. Since $t(x_i \leftarrow s) \in W_\tau(X)$ and $\mathcal{A}$ is complete, it follows that there is a state q, $q \in Q$ such that $\mathcal{A}(\gamma_0, s) = q$. Let $f_0 \in \mathcal{F}_0$ be such nullary operation symbol that $\Delta_0(f_0) = q$. Hence $\mathcal{A}(\gamma_0, t(x_i \leftarrow s)) = \mathcal{A}(\gamma_0, t(x_i \leftarrow f_0))$. Now, as in Lemma 1 we will obtain $x_i \in Ess(t, \mathcal{A})$ which is a contradiction. ■

Definition 3.3. A set $Y \subseteq Ess(t, \mathcal{A})$ is called *separable* for t and $\mathcal{A}$ w.r.t. a set $Z \subseteq Ess(t, \mathcal{A})$, with $Z \cap Y = \emptyset$ if there is an assignment γ on Z such that $Y \subseteq Ess(\gamma(t), \mathcal{A})$.

The set of all separable sets for t and $\mathcal{A}$ w.r.t. Z will be denoted by $Sep(t, \mathcal{A}, Z)$. When Y is separable for t and $\mathcal{A}$ w.r.t. $Z = Ess(t, \mathcal{A}) \setminus Y$ the set Y is called *separable* for t and $\mathcal{A}$ and the set of such Y will be denoted by $Sep(t, \mathcal{A})$.
When a set of essential inputs is not separable, it will be called *inseparable*.

Theorem 1. If $Y \in Sep(t, \mathcal{A})$ then for every input $x_i \in Y$ there exists at least one strong chain $x_i = t_k \lhd t_{k-1} \lhd \ldots \lhd t_1 \unlhd t$ such that $x_i \in Ess(t_j, \mathcal{A})$ for $j = 1, \ldots, k$.

The proof of the theorem can be done as Theorem 1 in [9].

Theorem 2. If $\mathcal{A}(\gamma, t_1) = \mathcal{A}(\gamma, t)$ for every $\gamma \in Ass(X, \mathcal{F}_0)$ then $Sep(t, \mathcal{A}) = Sep(t_1, \mathcal{A})$.

Proof. Let $Y \in Sep(t, \mathcal{A})$ and $Y = \{x_1, \ldots, x_m\}$. There is an assignment $\gamma_0 \in Ass(Z, \mathcal{F}_0)$, $Z = X \setminus Y$, such that $Y = Ess(\gamma_0(t), \mathcal{A})$. We have to prove that $Y \subseteq Ess(\gamma_0(t_1), \mathcal{A})$. Let $x_i \in Y$ be an arbitrary input variable from Y. It follows that there are two assignments $\gamma_1, \gamma_2 \in Ass(X, \mathcal{F}_0)$ with

$$\forall x_j \notin Y \quad \gamma_1(x_j) = \gamma_2(x_j) = \gamma_0(x_j), \quad \forall x_j \in Y,\ j \neq i \quad \gamma_1(x_j) = \gamma_2(x_j)$$

and $(\gamma_1(x_i) \neq \gamma_2(x_i))$ such that $\mathcal{A}(\gamma_1, t) \neq \mathcal{A}(\gamma_2, t)$. Hence $\mathcal{A}(\gamma_1, t_1) = \mathcal{A}(\gamma_1, t) \neq \mathcal{A}(\gamma_2, t) = \mathcal{A}(\gamma_2, t_1)$ i.e. $x_i \in Ess(\gamma_0(t_1), \mathcal{A})$. Consequently $Sep(t, \mathcal{A}) \subseteq Sep(t_1, \mathcal{A})$.
The inclusion $Sep(t_1, \mathcal{A}) \subseteq Sep(t, \mathcal{A})$ can be proved in a similar way. ■
The following lemma is obvious.

Lemma 3. If $Y \notin Sep(t, \mathcal{A}, Z)$ and $V \subset Ess(t, \mathcal{A})$ with $V \cap Z = \emptyset$ then $Y \cup V \notin Sep(t, \mathcal{A}, Z)$.

Further, we want to describe what the relation between separable sets for t and $\mathcal{A}$ and the "speed of runs" of $\mathcal{A}$ on t is?

Let us consider the following two transformations of t, depending on $\mathcal{A}$:
(i) if x_i is fictive for t and $\mathcal{A}$ and $f_0 \in \mathcal{F}_0$ then as result we obtain the tree $t' = t(x_i \leftarrow f_0)$;
(ii) if $t_1 \lhd t_2 \unlhd t$ with $\mathcal{A}(\gamma, t_1) = \mathcal{A}(\gamma, t_2)$ for each assignment $\gamma \in Ass(X, \mathcal{F}_0)$ then as result we have $t' = t(t_2 \leftarrow t_1)$.

When t' is an image of t under such a transformation we will write $t \vdash_{\mathcal{A}} t'$. The transitive closure of $\vdash_{\mathcal{A}}$ in $W_\tau(X)$ will be denoted by $\models_{\mathcal{A}}$.

Theorem 3. For every two terms t and s if $t \models_{\mathcal{A}} s$ then $\mathcal{A}(\gamma, t) = \mathcal{A}(\gamma, s)$ for every assignment $\gamma \in Ass(X, \mathcal{F}_0)$.

Proof. Let $t \vdash_{\mathcal{A}} s$. If $Dept(t) = 0$ then $t = x_i$ or $t = f_0$ for some $f_0 \in \mathcal{F}_0$. Clearly $s = t$ and the theorem is proved in this case. Let $Depth(t) \geq 1$. At first let s be a term obtained through applying a transformation with $t_2 \in X$. Hence $t = f(t_1, \ldots, t_n)$, with $x_i \notin Ess(t, \mathcal{A})$. Let $t_{i_1}, \ldots, t_{i_k}$ be all subterms amongs $t_1, \ldots, t_n$ for which $x_i \in Var(t_{i_p})$, $p = 1, \ldots, k$. Then $s = f(t_1, \ldots, t'_{i_1}, \ldots, t'_{i_k}, \ldots, t_n) = t(t_{i_1} \leftarrow t'_{i_1}, \ldots, t_{i_k} \leftarrow t'_{i_k}) = t(x_i \leftarrow f_0)$ where $t'_{i_p} = t_{i_p}(x_i \leftarrow f_0)$, $p = 1, \ldots, k$ for some $f_0 \in \mathcal{F}_0$. Hence for all $\gamma_1, \gamma_2 \in Ass(X, \mathcal{F}_0)$ if $\gamma_1(x_j) = \gamma_2(x_j)$ with $j \neq i$ then $\mathcal{A}(\gamma_1, t) = \mathcal{A}(\gamma_2, t)$. Let $\gamma \in Ass(X, \mathcal{F}_0)$ be an arbitrary assignment and let us consider the assignment $\gamma' \in Ass(X, \mathcal{F}_0)$ defined as follows: $\gamma'(x) = f_0$ if $x = x_i$ and $\gamma'(x) = \gamma(x)$ if $x \neq x_i$. Thus we have $\mathcal{A}(\gamma', t) = \mathcal{A}(\gamma, t)$ and $\mathcal{A}(\gamma', t) = \mathcal{A}(\gamma, t(x_i \leftarrow f_0)) = \mathcal{A}(\gamma, s)$. The theorem is proved in this case.

Let s be a term obtained through applying a transformation with t_2, $Depth(t_2) > 0$. Hence there are subterms $t_1 \lhd t_2 \unlhd t$ with $\mathcal{A}(\gamma, t_1) = \mathcal{A}(\gamma, t_2)$ for every $\gamma \in Ass(X, \mathcal{F}_0)$ and $s = t(t_2 \leftarrow t_1)$. Clearly $\mathcal{A}(\gamma, s) = \mathcal{A}(\gamma, t(t_2 \leftarrow t_1)) = \mathcal{A}(\gamma, t(t_2 \leftarrow t_2)) = \mathcal{A}(\gamma, t)$. ∎

4. Complexity of Automata on Trees

It is easy to see that if $t \lhd s$ with $\mathcal{A}(\gamma, t) = \mathcal{A}(\gamma, s)$ for each assignment $\gamma \in Ass(X, \mathcal{F}_0)$ then the results of the runs of $\mathcal{A}$ on t and s will be the same, but the run on t will be "quicker" than the run on s because of $t \lhd s$. So, we need a definition of the "quickness" of runs of an automaton on a tree.

Let t be a tree and $\mathcal{A}$ be an automaton. The set of all states of $\mathcal{A}$ which can be associated with t will be denoted by $St(t, \mathcal{A})$ and $st(t, \mathcal{A}) = |St(t, \mathcal{A})|$ is the number of the elements in $St(t, \mathcal{A})$. Thus $q \in St(t, \mathcal{A})$ if and only if there is an assignment $\gamma \in Ass(X, \mathcal{F}_0)$ such that $\mathcal{A}(\gamma, t) = q$.

Definition 4.1. The *complexity* of $\mathcal{A}$ on t denoted by $Comp(t, \mathcal{A})$ is defined in the following inductive way:
(i) If $t = x \in X$ then $Comp(t, \mathcal{A}) = \sum_{f_0 \in \mathcal{F}_0} st(f_0, \mathcal{A})$;
(ii) If $t = f_0 \in \mathcal{F}_0$ then $Comp(t, \mathcal{A}) = st(f_0, \mathcal{A})$;
(iii) If $t = f(t_1, \ldots, t_n)$ then

$$Comp(t, \mathcal{A}) = \prod_{j=1}^{n} st(t_j, \mathcal{A}) + \sum_{i=1}^{n} Comp(t_i, \mathcal{A}).$$

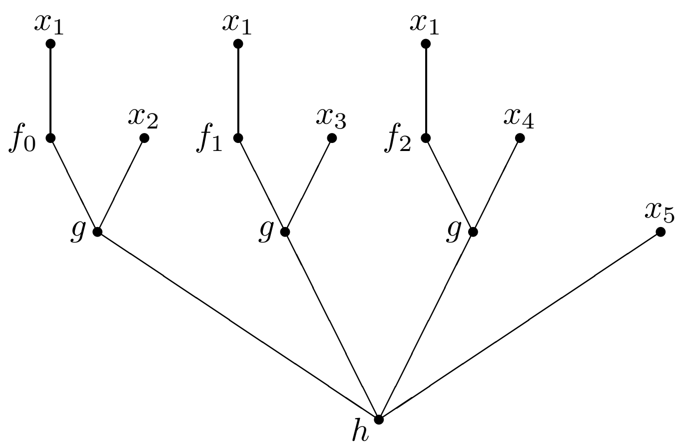

FIGURE 1. Tree of a term

If automaton $\mathcal{A}$ is complete and deterministic then $Comp(x, \mathcal{A}) = |\mathcal{F}_0|$ and $Comp(f_0, \mathcal{A}) = 1,\ f_0 \in \mathcal{F}_0$.

So, the complexity of $\mathcal{A}$ on t presents the number of all calculations of values of Δ for all runs of $\mathcal{A}$ on t.

It is clear that if $t \models_{\mathcal{A}} s$ then $Comp(s, \mathcal{A}) \leq Comp(t, \mathcal{A})$.

Example 1. Let $\mathcal{A} = \langle Q, \mathcal{F}, X, Q_f, \Delta\rangle$ with $\mathcal{F}_0 = \{0, 1, 2\}$, $\mathcal{F}_1 = \{f_0, f_1, f_2\}$, $\mathcal{F}_2 = \{g\}$, $\mathcal{F}_4 = \{h\}$, $Q = \{q_0, q_1, q_2\}$, $Q_f = \{q_1\}$, $\Delta_0(i) = q_i$ for $i = 0, 1, 2$, $\Delta_1(f_i, q_j) = \begin{cases} q_1, & \text{if} \quad i = j \\ q_0, & \text{if} \quad i \neq j; \end{cases}$ for $i = 0, 1, 2$, $\Delta_2(g, q_i, q_j) = q_m$, where $m = i.j\ (mod\ 3)$ and $\Delta_4(g, q_i, q_j, q_k, q_l) = q_m$, where $m = i + j + k + l\ (mod\ 3)$.
Let us consider the term $t = h(g(f_0(x_1), x_2), g(f_1(x_1), x_3), g(f_2(x_1), x_4), x_5)$, with the tree, given on the Figure 1

The subterms of this term are: $t_1 = g(f_0(x_1), x_2)$, $t_2 = g(f_1(x_1), x_3)$, $t_3 = g(f_2(x_1), x_4)$, $t_4 = x_5$, $t_{11} = f_0(x_1)$, $t_{12} = x_2$, $t_{21} = f_1(x_1)$, $t_{22} = x_3$, $t_{31} = f_2(x_1)$, $t_{32} = x_4$, $t_{111} = x_1$, $t_{211} = x_1$, $t_{311} = x_1$.
Let us calculate $Comp(t, \mathcal{A})$. Clearly
$Comp(t_{111}, \mathcal{A}) = Comp(t_{211}, \mathcal{A}) = Comp(t_{311}, \mathcal{A}) = Comp(t_{12}, \mathcal{A}) =$
$= Comp(t_{22}, \mathcal{A}) = Comp(t_{32}, \mathcal{A}) = Comp(t_4, \mathcal{A}) = 3$. Because $f_i \in \mathcal{F}_1$ and $st(x_1, \mathcal{A}) = 3$ it follows that $Comp(f_i(x_1), \mathcal{A}) = 6$ for $i = 0, 1, 2$, i.e. $Comp(t_{11}, \mathcal{A}) = Comp(t_{21}, \mathcal{A}) = Comp(t_{31}, \mathcal{A}) = 6$, Let us note that $St(t_{i1}, \mathcal{A}) = \{q_0, q_1\}$ for $i = 1, 2, 3$ and $st(t_{i1}, \mathcal{A}) = 2$ for $i = 1, 2, 3$. Analogously, $st(t_{i2}, \mathcal{A}) = 3$ for $i = 1, 2, 3$. Thus $Comp(t_i, \mathcal{A}) = 2.3 + 6 + 3 = 15$ for $i = 1, 2, 3$ i.e. $Comp(t_1, \mathcal{A}) = Comp(t_2, \mathcal{A}) = Comp(t_3, \mathcal{A}) = 15$. It is easy to see that $st(t_i, \mathcal{A}) = 3$ for $i = 1, 2, 3, 4$. Hence $Comp(t, \mathcal{A}) = 3.3.3.3 + 15 + 15 + 15 + 3 = 129$.

5. Distributive Sets of Inseparable Sets of Inputs

We will consider the case when a set of essential inputs is inseparable. It seems that if a term has "many" inseparable sets the runs of $\mathcal{A}$ on such a term will be "quicker".

Definition 5.1. Let $Y, Z \subseteq Ess(t, \mathcal{A})$, $\quad Y \cap Z = \emptyset$ and $Y \notin Sep(t, \mathcal{A})$. The set Z is called *distributive set* of Y for t and $\mathcal{A}$ if $Y \not\subseteq Ess(\gamma(t), \mathcal{A})$ for every $\gamma \in Ass(Z, \mathcal{F}_0)$ and Z is minimal with respect to this property.

The family of all distributive sets of Y will be denoted by $Dis(Y, t, \mathcal{A})$. Note that the family of distributive sets of Y is non-empty iff Y is not separable.

Theorem 4. If $Z \in Dis(Y, t, \mathcal{A})$ then for each proper subsets Z_1 and Y_1 of Z and Y it is held that $Z_1 \notin Dis(Y_1, t, \mathcal{A})$.

Proof. Let Y_1 is a proper subset of Y. Suppose the theorem is false and let Z_1 is a proper subset of Z with $Z_1 \in Dis(Y_1, t, \mathcal{A})$. Because of Lemma 3 it follows that $Z_1 \in Dis(Y, t, \mathcal{A})$. This contradicts to the minimality of Z as a distributive set of Y and $\mathcal{A}$. ∎

The next example is a good illustration of how to use distributive sets to obtain "quicker" runs of $\mathcal{A}$ on t under different assignments.

Example 2. Let us try to find a simpler way for running of $\mathcal{A}$ on t and $\gamma \in Ass(X, \mathcal{F}_0)$ where t and $\mathcal{A}$ are as in Example 1.

Let $Y = \{x_2, x_3, x_4\}$, $Z = \{x_1\}$ and $\gamma \in Ass(Z, \mathcal{F}_0)$. There are only the following three possible cases.
a) If $\gamma(x_1) = 0$ then $x_3, x_4 \notin Ess(\gamma(t), \mathcal{A})$;
b) if $\gamma(x_1) = 1$ then $x_2, x_4 \notin Ess(\gamma(t), \mathcal{A})$;
c) if $\gamma(x_1) = 2$ then $x_2, x_3 \notin Ess(\gamma(t), \mathcal{A})$.
Hence $Y \notin Sep(t, \mathcal{A})$ and $Z \in Dis(Y, t, \mathcal{A})$.
Now, we can consider $Comp(t, \mathcal{A})$ and use distributive set Z to obtain simpler runs of $\mathcal{A}$ on t. The fact that Z is a distributive set of Y allows us to distribute all 243 assignments in three classes $\Gamma_0, \Gamma_1, \Gamma_2$ according to a),b) and c) i.e. $\gamma \in \Gamma_i \iff \gamma(x_1) = i$. Let $\gamma \in Ass(X, \mathcal{F}_0) \cap \Gamma_0$. We can apply a transformation defined as above on the tree $\gamma(t) = h(g(f_0(0), x_2), g(f_1(0), x_3), g(f_2(0), x_4), x_5)$. By $\Delta_1(f_i, q_j) = 0$ when $i \neq j$ it follows that $\gamma(t) \models_{\mathcal{A}} s_0$, where $s_0 = h(x_2, 0, 0, x_5)$ (see Figure 2). It is easy to calculate $Comp(s_0, \mathcal{A}) = 17$. In an analogous way the trees s_i (see Figure 2) when $\gamma \in Ass(X, \mathcal{F}_0) \cap \Gamma_i$, $\; i = 1, 2$ with $Comp(s_i, \mathcal{A}) = 17$, $i = 1, 2$ can be obtained.

So, we have a very simple procedure to execute the runs of $\mathcal{A}$ on t with given $\gamma \in Ass(X, \mathcal{F}_0)$. This procedure consists of:
Step 1. Find i, $i \in \{0, 1, 2\}$ such that $\gamma \in \Gamma_i$.
Step 2. Find $\mathcal{A}(\gamma, s_i)$.

Note that step 1. can be realized by a simple checking $\gamma(x_1) = 0|1|2$. We can naturally assume that the complexity of this step equals 3. Thus the complexity of the whole procedure is 20 and in the general case it is 129.

This example is a good motivation for future investigations of the inseparable sets and their distributive sets.

Theorem 5. If $Z \in Dis(Y, t, \mathcal{A})$ then for each proper subsets Z_1 and Y_1 of Z and Y it is held that $Z_1 \notin Dis(Y_1, t, \mathcal{A})$.

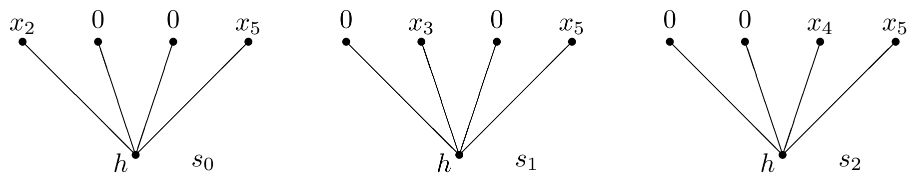

FIGURE 2. Distributed trees

Proof. Let Y_1 is a proper subset of Y. Suppose the theorem is false and let Z_1 is a proper subset of Z with $Z_1 \in Dis(Y_1, t, \mathcal{A})$. Because of Lemma 3 it follows that $Z_1 \in Dis(Y, t, \mathcal{A})$. This is a contradiction with the minimality of Z as a distributor of Y and $\mathcal{A}$. ■

Definition 5.2. Let $\mathcal{M} = \{M_1, \dots, M_m\}$ be a finite family of nonempty sets. A set $M = \{z_1, \dots, z_l\}$ is called *representative system* for $\mathcal{M}$ if $M \cap M_i \neq \emptyset$ for every $i \in \{1, \dots, m\}$ and M is minimal with respect to this property.

Lemma 4. If M is a representative system for $\mathcal{M}$ then the following is true:
(i) For each $M_i \in \mathcal{M}$ there is $z_j \in M$ with $z_j \in M_i$;
(ii) For each $z_j \in M$ there is $M_i \in \mathcal{M}$ with $\{z_j\} = M_i \cap M$.

Proof. The statement (i) is obvious. To prove (ii) let us suppose there is $z_j \in M$ with $\{z_j\} \neq M_i \cap M$ for every $M_i,\ M_i \in \mathcal{M}$.
Hence if $z_j \in M_i$ then $|M_i \cap M| \geq 2)$ for every $M_i,\ M_i \in \mathcal{M}$.
This means that $M \setminus \{z_j\}$ is a representative system for $\mathcal{M}$. A contradiction. ■

Theorem 6. Let $Y = \{x_1, \dots, x_k\} \notin Sep(t, \mathcal{A})$. If $Z = \{x_{k+1}, \dots, x_m\},\ k < m$ is a representative system for $Dis(Y, t, \mathcal{A})$ then $Y \cup Z \in Sep(t, \mathcal{A})$.

Proof. We will consider the non-trivial case $|Y| \geq 2$. Clearly $Dis(Y, t, \mathcal{A}) \neq \emptyset$. Let us set $V = \{x_{m+1}, \dots, x_n\} = Ess(t, \mathcal{A}) \setminus (Y \cup Z)$. Since, Z is representative system for $Dis(Y, t, \mathcal{A})$ it follows that $V_1 \notin Dis(Y, t, \mathcal{A})$ for each $V_1 \subseteq V$ and there is an assignment $\gamma \in Ass(V, \mathcal{F}_0)$ such that $Y \in Ess(\gamma(t), \mathcal{A})$.
We have to prove that $Z \subset Ess(\gamma(t), \mathcal{A})$. Suppose this is false. Without loss of generality assume that $x_{k+1} \notin Ess(\gamma(t), \mathcal{A})$.
Let $Z_1 = \{x_{k+1}, x_{j_1}, \dots, x_{j_l}\},\ j_l \leq n$ be a distributor of Y for t and $\mathcal{A}$ such that $Z_1 \cap Z = \{x_{k+1}\}$. The existence of Z_1 follows by Lemma 4. Thus we have
$\{x_{j_1}, \dots, x_{j_l}\} \subseteq V, \quad Ess(\gamma(t), \mathcal{A}) \cap \{x_{j_1}, \dots, x_{j_l}\} = \emptyset$ and $Ess(\gamma(t), \mathcal{A}) \cap Z_1 = \emptyset$.
Let $f_0 \in \mathcal{F}_0$ be an arbitrary nullary operation symbol and $\gamma_1 \in Ass(Z_1, \mathcal{F}_0)$ be an assignment defined as follows:

$$\gamma_1(x) = \begin{cases} f_0 & \text{if} \quad x = x_{k+1}; \\ \gamma(x) & \text{if} \quad x \in Z_1 \cap V. \end{cases}$$

Since $(Z_1 \setminus V) \cap Ess(\gamma(t), \mathcal{A}) = \emptyset$ it follows that $Ess(\gamma_1(t), \mathcal{A}) = Ess(\gamma(t), \mathcal{A})$. Consequently $Y \subset Ess(\gamma_1(t), \mathcal{A})$ and $Z_1 \notin Dis(Y, t, \mathcal{A})$. This is a contradiction. ■
There are examples showing that any representative system Z of the family of distributive sets of Y is a maximal set for which $Y \cup Z \in Sep(t, \mathcal{A})$ i.e. the Theorem 6 can not be generalized in this direction.

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